Non-Classical Operational Calculus in the Analysis of the Behavior of Geosynthetics in Railway Engineering Structures

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Abstract

One of the basic tasks of communication structures is to safely carry loads and then transfer them to the ground. In the case of railway surfaces, in addition to static loads, dynamic loads are of particular importance. In the discussed case, this involves adopting an appropriate model to describe the behavior of geosynthetics in railway engineering structures. When analyzing the cooperation of geosynthetics with elements of an engineering structure (e.g. subgrade, subsoil, pipeline), we can treat them, for example, as elastic membranes or coatings placed on various types of substrate. Modeling a real system means idealizing it in terms of taking into account those features of the medium and its elements that seem most important from the point of view of the analyzed problem. A physical model will be built, followed by a mathematical model using non-classical operational calculus. The mentioned operational calculus is not based, as is the case with the classical operator calculus, on integral transformations (Laplace, Laplace - Carson, Fourier, Mellin, Hankel, etc.). Therefore, it is freed from many limiting assumptions required in the case of classical operational calculus. The non-classical operational calculus is based on three arbitrary linear operations S, T, s and two arbitrary linear spaces L^0, L^1. This allows us to create continuous and discrete models and analyze them in a uniform way. In the mathematical description of the behavior of geosynthetics in railway engineering structures in the continuous domain, the Laplace operator $\triangle = \nabla^2$ and the d'Alembert operator $\square$ are used. The d'Alembert operator is treated as an operation S, for which operations T, s are selected and the location of points P from the geosynthetic at time t is determined. Assuming S as an operation defined in the space of multi-point sequences allows us to analyze the formulated problem in the discrete domain. Such approach to the analyzed problem allows us to determine the location of P points from the geosynthetic at time t using analytical, numerical or hybrid methods.

Keywords: *geosynthetics, railway track, trackbed strengthening, trackbed modeling*

1. Introduction

Geosynthetics, i.e. geotextiles, geotextiles, geogrids, geomembranes, geolinings, etc., are widely used in construction, including the construction, modernization and renovation of various transport lines, transmission lines (pipelines), communication lines (railroads [1] - see Fig. 1 in [2] by E. Mieloszyk and S. Grulkowski, roadways, etc.). Linear transport structures are multi-layered, so each subsequent layer is characterized by different load conditions (see Fig. 2 in [2] by E. Mieloszyk and S. Grulkowski).

As a result of the loads acting on the railway road during its current operation, it settles, often unevenly (see Fig. 1 in [3] by E. Mieloszyk and S. Grulkowski and A. Milewska).

This means that points lying on the geosynthetic also change their position in time - see also Fig. 1.

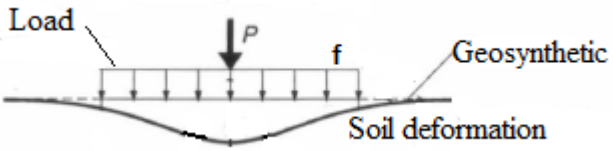

Fig. 1. Example of changing the position of points on the geosynthetic.

In each case, when analyzing the behavior of geosynthetics, we can treat them as elastic membranes or coatings placed on various types of substrate. For example, Winkler, Euler, Pasternak, Kerr type, because modeling a real system means its idealization in terms of taking into account those features of the medium that seem most important from the point of view of the analyzed problem.

2. Elements of modeling the behavior of geosynthetics

The constructed physical and then mathematical model is usually presented in the form of differential or difference equations. In the form of ordinary or partial differential equations, or their systems with constant or variable coefficients [4]. The Laplace operator is often used in this description (1).

$$\triangle = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1}$$

leading to a more general operator, the d'Alembert operator (2)

$$\square = \triangle - \frac{1}{a^2}\frac{\partial^2}{\partial t^2}, a \neq 0. \tag{2}$$

Similarly, a slab on a ground base (elastic base) is described using the bilaplasian Δ^2 [5]. Similarly (in the sense of differential problems), slab systems or foundations are modeled [6], slab-pile foundations [1, 7, 8, 9] by selecting appropriate system parameters taking into account the parameters of the subsoil [10]. Railway lines, tram lines, pipelines and runways are also built on slabs and slab-pile foundations. Geosynthetics are also used in these cases. In [11], the authors use an elastic substrate in modeling to assess the impact of uneven settlement of the tank bottom on the distribution of internal forces at its bottom. This is important for determining the failure rate of large tanks [12]. Such concepts are used in the analysis of the mutual interaction in the track - railway vehicle - trackbed - subgrade system, including their experimental verification [13], also by identifying the subgrade, e.g. using CPTU type probing (Fig. 2) and determining its parameters [14].

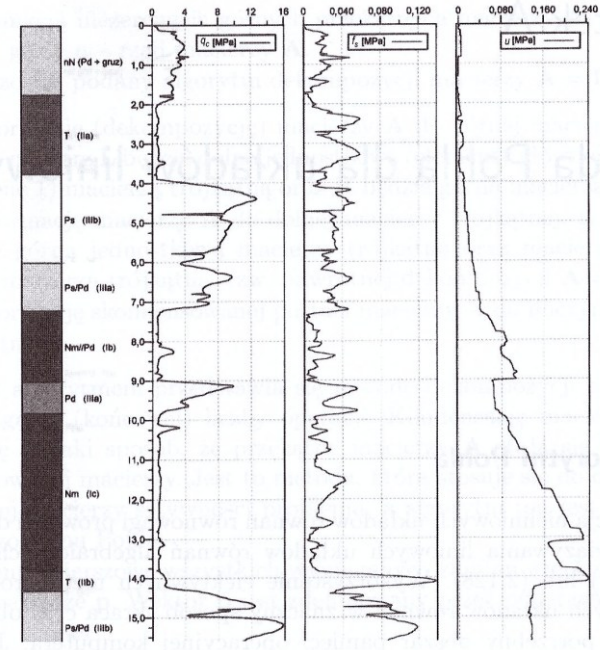

Fig. 2. An example of the CPTU test result at a selected probing point (node).

Note that we are dealing here with analog arrangements. Therefore, to analyze the problem of cooperation, it is worth using mathematical methods that allow for a uniform description and analysis of these various situations. Such possibilities are created by the generalized Taylor formula [15], based on the non-classical operational calculus [15, 16]. It allows, for example, to analytically express deformations, e.g. by spherical waves with the assumed equivalent load and ground parameters, allows for approximation, estimation of the approximation error and expression of the solution to the problem in operator form using the generalized Heaviside operator. It also allows for numerical analysis of the problem using mixed methods.

3. On the non-classical operator calculus

To achieve the set goal, a non-classical operational calculus will be used. It is not based, as is the case with the classical operational calculus, on integral transformations (Laplace, Laplace-Carson, Mellin, Hankel, etc.). Therefore, it is freed from many limiting assumptions required in the case of classical operational calculus. It entered the classification lists of mathematics departments. In the Mathematical Reviews ® Mathematical Subject Classification (MSC2010) it is listed under number 44 Integral Transforms, Operational Calculus and further detailed under number 44A40 – Calculus of Mikusinski and other operational calculi.

Non-classical operational calculus allows for a uniform description and analysis of various types of engineering (mechanical) problems in the continuous or discrete domain, including methods of solving the relevant differential or difference equations and their systems.

The non-classical operational calculus is based on three arbitrary linear operations S, T, s and two arbitrary linear spaces L^0, L^1.

We assume that

$$S: L^1 \to L^0,$$
$$T: L^0 \to L^1, L^1 \subset L^0,$$
$$s: L^1 \to KerS = \{x \in L^1: Sx = 0\}$$

and additionally that

$$ST = id, \tag{3}$$

$$TS = id - s, \tag{4}$$

(The identity operator is denoted by $\boldsymbol{id}$.)

Properties of $\boldsymbol{S, T, s}$ operations and their various examples can be found in [15, 16]. Importantly, the mentioned operations can be defined in the continuous or discrete domain and then they lead to a description using differential equations, difference equations or their systems, and can be used to describe and analyze generalized dynamic systems [15]. In the continuous domain,

linear spaces $\boldsymbol{L^0}, \boldsymbol{L^1}$ are functional spaces of one or more variables. In the discrete domain, linear spaces $\boldsymbol{L^0}, \boldsymbol{L^1}$ are spaces of single-index or multi-index sequences.

It turns out that to analyze our problem, we should assume the Laplace operator $\triangle$ or the d'Alembert operator $\square$ as the operation S and select the operations T and s to them. In the case of $\square$, these are the following operation.

$$T\{f(P,t)\} = \left\{-\frac{1}{4\pi}\iiint_{\Omega}\frac{Af(P_0,t)}{d(P,P_0)}d\Omega\right\} \tag{5}$$

$$s\{u(P,t)\} = \left\{\frac{1}{4\pi}\iint_{\sigma}\left[\frac{1}{d(P,P_0)}A\left(\frac{\partial u}{\partial n}\right) - Au\frac{\partial}{\partial n}\left(\frac{1}{d(P,P_0)}\right) + \frac{1}{ad(P,P_0)}A\left(\frac{\partial u}{\partial t}\right)\frac{\partial d(P,P_0)}{\partial n}\right]d\sigma\right\} \tag{6}$$

where: $L^0 = \mathrm{C}^1\big(\Omega \times< 0,\infty)\big)$, $L^1 = \mathrm{C}^3(\Omega \times< 0,\infty))$, operation A is the substitution of t values $\mathrm{t}_0 - \frac{\mathrm{d(P,P_0)}}{\mathrm{a}}$, $\mathrm{a} \neq 0$ is a constant characterizing the equivalent system, while $\mathrm{d(P,P_0)} = \sqrt{(\mathrm{x}-\mathrm{x_0})^2 + (\mathrm{y}-\mathrm{y_0})^2 + (\mathrm{z}-\mathrm{z_0})^2}$, and $\frac{\partial}{\partial \mathrm{n}}$ denotes the normal derivative, σ is the boundary of the set Ω.

Of course, the operations T and s defined by formulas (5) and (6), respectively, together with the operation S=$\square$, meet important conditions (3) and (4).

Our mathematical modeling involves a partial differential equation

$$\square u = f(x,y,z,t) \tag{7}$$

It describes an equivalent system in which the constant a characterizes the subsoil (base) with geosynthetics (Fig. 2), while *f(x,y,z,t)* characterizes a continuous equivalent load, also depending on the time *t* of this load. These values must be determined empirically [13, 14, 17] or approximated by the parameters of the system components, which in the model can be treated as connected in series or parallel and may create a substitute dynamic system [4, 15].

Of course, *x, y, z* are the coordinates of the point $\mathrm{P} \in \Omega$ at time *t* and *u* is a function of the appropriate class defined in the set $\Omega \times< 0,\infty)$ and means deflection (displacement, change in the position of points *P*). In our model, it is enough that Ω is a prism or, in particular, a cuboid with appropriate dimensions. We will want to determine the values of the function *u* at point *P* located inside the set Ω at time *t*.

Using the formulas for *T* and *s* and the generalized Taylor formula [15], we can write

$$u(x,y,z,t) = u(P,t) = \left\{\frac{1}{4\pi}\iint_{\sigma}\left[\frac{1}{d(P,P_0)}A\left(\frac{\partial u}{\partial n}\right) - Au\frac{\partial}{\partial n}\left(\frac{1}{d(P,P_0)}\right) + \frac{1}{ad(P,P_0)}A\left(\frac{\partial u}{\partial t}\right)\frac{\partial d(P,P_0)}{\partial n}\right]d\sigma\right\} + \left\{-\frac{1}{4\pi}\iiint_{\Omega}\frac{Af(P_0,t)}{d(P,P_0)}d\Omega\right\} \tag{8}$$

The last formula (8) shows that if $\mathrm{u_1}$ is the response of a free system $\square u$=0, i.e. it is a spherical wave, and $\mathrm{u_2}$ is the response of a non-free system with forcing (load) *f(x,y,z,t)*, then the response *u* of this system can be expressed as $\mathrm{u} = \mathrm{u_1} + \mathrm{u_2}$.

If $\mathrm{d(P,P_0)} > R$ i and the equivalent load is a bounded function, then the second term of the last sum can be estimated by $\frac{\mu(\Omega)\sup|\mathrm{f}|}{4\pi\mathrm{R}}$, where $\mu(\Omega)$ is a measure of the set Ω. Then we get this

$$u(P,t) \leq \left\{\frac{1}{4\pi}\iint_{\sigma}\left[\frac{1}{d(P,P_0)}A\left(\frac{\partial u}{\partial n}\right) - Au\frac{\partial}{\partial n}\left(\frac{1}{d(P,P_0)}\right) + \frac{1}{ad(P,P_0)}A\left(\frac{\partial u}{\partial t}\right)\frac{\partial d(P,P_0)}{\partial n}\right]d\sigma\right\} + \frac{\mu(\Omega)\sup|\mathrm{f}|}{4\pi R}. \tag{9}$$

This means that the change in the position of the points of the analyzed substrate, including points from geosynthetics, will not exceed the right side of the last formula (9), i.e. they can be estimated by spherical functions (spherical waves).

4. Discretization of the problem

If we accept.

$$\mathrm{x} = \mathrm{i}\Delta\mathrm{x}\,, \qquad \mathrm{y} = \mathrm{j}\Delta\mathrm{y}\,, \qquad \mathrm{z} = \mathrm{k}\Delta\mathrm{z}\,, \qquad \mathrm{t} = \mathrm{l}\Delta\mathrm{t}$$

where $\Delta\mathrm{x}, \Delta\mathrm{y}, \Delta\mathrm{z}, \Delta\mathrm{t}$ are the increments of the variables, respectively $\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{t}$
and we denote by

$$\mathrm{u_{i,j,k,l}} = \mathrm{u(i\Delta x, j\Delta y, k\Delta z\,, l\Delta t)} \quad , \qquad \mathrm{f_{i,j,k,l}} = \mathrm{f(i\Delta x, j\Delta y, k\Delta z\,, l\Delta t)}\,,$$

where *i, j, k, l* are natural numbers, then we can move from continuous analysis to discrete analysis.

By replacing the partial derivatives with their difference quotients and introducing *S* operations in the space of four-point discrete signals

$$\begin{gathered}\mathrm{S_1 u_{i,j,k,l}} = \mathrm{u_{i,j,k,l}}\,,\\ \mathrm{S_2 u_{i,j,k,l}} = \mathrm{u_{i,j+1,k,l}}\,,\\ \mathrm{S_3 u_{i,j,k,l}} = \mathrm{u_{i,j,k+1,l}}\,,\\ \mathrm{S_4 u_{i,j,k,l}} = \mathrm{u_{i,j,k,l+1}}.\end{gathered}$$

or

$$\begin{gathered}\mathrm{S_1 u_{i,j,k,l}} = \mathrm{u_{i+1,j,k,l} - u_{i,j,k,l}}\,,\\ \mathrm{S_2 u_{i,j,k,l}} = \mathrm{u_{i,j+1,k,l} - u_{i,j,k,l}},\\ \mathrm{S_3 u_{i,j,k,l}} = \mathrm{u_{i,j,k+1,l} - u_{i,j,k,l}}\,,\\ \mathrm{S_4 u_{i,j,k,l}} = \mathrm{u_{i,j,k,l+1} - u_{i,j,k,l}}.\end{gathered}$$

we can replace the analyzed continuous dynamic system (7), i.e.

$$\square\mathrm{u} = \mathrm{f(x,y,z,t)}$$

by discrete dynamic systems with distributed parameters. They can be used to study, in the field of four-index discrete signals, a continuous system or they can be used to numerically determine the response of a continuous model.

5. Conclusion

The presented considerations allow us to use the created mathematical descriptions to model changes in the position of points in the analyzed area Ω over time, and in particular the points of interest to us *(x,y,g(x,y))* of geosynthetics, where *z=g(x,y)* is the surface equation of the geosynthetic, while *(x,y)* belong to the set $D \subset R^2$, which is the projection Ω onto the XOY plane of the coordinate system and points lying in layers located sufficiently close to the surface of the geosynthetic.

The quantities *u(x,y,z,t)* can be determined numerically, or using mixed methods based on the presented formulas, or FEM shell models can be used. They can be used for simulation and experimental research.

In order to identify the model, it is necessary to conduct experimental research using an active, passive or mixed experiment and validate the model.

Using the obtained formulas, it is possible to conduct a quantitative and qualitative analysis of the problem, after identification, calibration and validation of the model, preferably using in situ tests to determine the parameters of the equivalent system - e.g. probing.

The analysis of the problem is further complicated by the interaction of individual elements of the railway structure. E.g. railway surface, trackbed with geosynthetics and subsoil. However, a solution to this situation may be the use of analog and substitute arrangements.

The use of non-classical operator calculus and the related generalized Taylor formula proved helpful throughout the analysis. This leads to continuous and discrete models also aimed at numerically solving the problem.

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