# Functionally Graded Shafts under Torsion: An Analytical Approach for Investigation of Damping

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*Abstract*

*The question for the damping energy in functionally graded viscoplastic shafts loaded in cyclic torsion is addressed in the present paper. In particular, a circular shaft is researched analytically here. It should be specified that the prior damping studies deal mainly with rectangular beams loaded in cyclic bending. However, circular shafts also are widely applied in various load-bearing constructions in the current engineering. Therefore, there is a need of researching damping in circular shafts under cyclic torsion. The shafts researched here are functionally graded. The necessity of studying functionally graded structures is conditioned by quickly increasing use of functionally graded materials and structures in a variety of engineering applications. Studying of damping problem has meaning for guarantying the safety and reliability of structures. Both ends of the shaft analyzed in the present paper are rigidly fixed. The torsion is induced by a moment applied near the mid-span. The damping analysis uses a Ramberg-Osgood type of stress-strain-time relationship. The material parameters of the stress-strain-time relationship are distributed continuously along the cross-section radius. The unit damping energy parameters and their variations along the radius of the shaft cross-section are explored. The damping energy in the shaft is found-out by integrating of unit damping energy. The solution derived offers very good possibilities for obtaining various numerical results with purpose of clarifying how the damping energy is influenced by different factors such as distribution of material parameters along the radius of the cross-section, geometry of the functionally graded shaft, and parameters of the external loading.*

**Keywords:** *Viscoplastic behavior, Functionally graded material, Damping, Shaft*

## 1. Introduction

One of the motives for conducting the present analytical research is that the functionally graded materials (FGM) are commonly used in modern engineering for manufacturing of various components and members of constructions [1- 4]. This can be explained mainly by the fact that in many practical applications the FGM are more suitable than the traditional engineering materials [5-10]. As known, the properties of FGM change smoothly, i.e. the properties are smooth functions of coordinates in a structural member or in a component manufactured by using a FGM [11-15]. Another motive for conducting this research is that the previous studies of damping deal mainly with structures of rectangular cross-section subjected to bending [16-18].

Very often the functionally graded members and components of various structures are subjected to cyclic loading. Under such loading conditions, the question for analysis of damping is of prime significance. The analysis of damping is closely related to the problems of safety and reliability of the engineering constructions and equipments.

The present paper researches damping energy in a shaft made of a FGM with viscoplastic mechanical behavior. The shaft is under cyclic torsion. The distribution of material parameters along the cross-section radius is treated by using exponential functions. The analysis of damping goes through researching the shaft with considering the viscoplastic behavior. In the context of analyzing of the shaft, relevant equations of equilibrium are compiled and solved. The damping is found-out by integrating the unit damping energy in the shaft. Numerical results illustrating how the damping is influenced by various factors are obtained and presented in form of graphs.

## 2. Approach for Analyzing the Damping Energy

We consider a viscoplastic shaft of length, $l$, as shown in Fig. 1. The shaft is functionally graded along the radius, $R_B$, of the cross-section. Furthermore, the shaft is loaded in cyclic torsion with torsion moment, $T_a$, about zero mean. The torsion moment acts in section, $H_2$, standing at distance, $l_1$, from end section, $H_1$, as indicated in Fig. 1. The sections, $H_1$ and $H_3$, are rigidly fixed (thus, the shaft is a system having one degree of static indeterminacy).

We use a stress-strain-time relationship of the type of Ramberg-Osgood for describing the viscioplastic behavior of the shaft in the current theoretical analysis of the damping. The stress-strain-time relation is given by [17]

$$\gamma_a = \frac{\tau_a}{G} + \left(\frac{\tau_a}{H_C}\right)^{\frac{1}{n_C}}, \tag{1}$$

where

$$n_C = \frac{1}{\delta}, \tag{2}$$

$$H_C = \frac{1}{\left(D_3 t^{\phi}\right)^{\frac{1}{\delta}}}. \tag{3}$$

In the above formulas, $t$ is time, $G$ is the shear modulus, $\gamma_a$ is the shear strain, $\tau_a$ is the shear stress, $\phi$, $D_3$ and $\delta$ are material parameters.

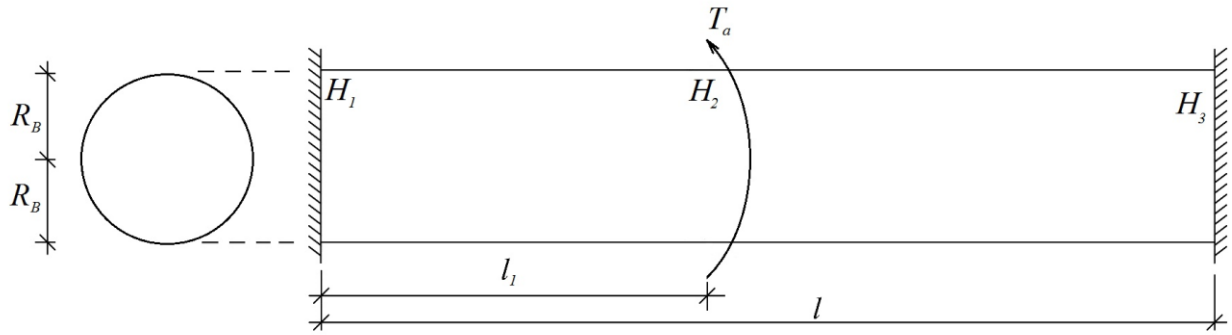


Fig. 1. Geometry and loading of a functionally graded shaft.

We use the following formulas for presenting the continuous change of $G$, $\phi$, $D_3$ and $\delta$ along the cross-section radius:

$$G = G_Q e^{m\frac{R}{R_B}}, \tag{4}$$

$$\phi = \phi_Q e^{b\frac{R}{R_B}}, \tag{5}$$

$$D_3 = D_{3Q} e^{p\frac{R}{R_B}}, \tag{6}$$

$$\delta = \delta_Q e^{s\frac{R}{R_B}}, \tag{7}$$

where

$$0 \le R \le R_B. \tag{8}$$

In formulas (4), (5), (6) and (7), $G_Q$, $\phi_Q$, $D_{3Q}$ and $\delta_Q$ are the values of $G$, $\phi$, $D_3$ and $\delta$ in the cross-section centre, $m$, $b$, $p$ and $s$ are parameters which regulate the change of $G$, $\phi$, $D_3$ and $\delta$, respectively.

The analysis of the damping energy in the shaft is related to integration of the unit damping energy, $\Delta u$. We apply the following formula for determining $\Delta u$ [17]:

$$\Delta u = \frac{4\left(1-n_C\right)\tau_a^{1+\frac{1}{n_C}}}{\left(1+n_C\right)\left(H_C\right)^{\frac{1}{n_C}}}. \tag{9}$$

For the damping energy, $\Delta U$, we get

$$\Delta U = \Delta U_{H1H2} + \Delta U_{H2H3}, \tag{10}$$

where

$$\Delta U_{H1H2} = l_1 \iint\limits_{(A)} \Delta u_{H1H2} dA, \tag{11}$$

$$\Delta U_{H2H3} = (l - l_1) \iint\limits_{(A)} \Delta u_{H2H3} dA. \tag{12}$$

In formulas (11) and (12), $A$ is the cross-section area, $\Delta u_{H1H2}$ and $\Delta u_{H2H3}$ are the unit damping energies in the two sectors of the shaft.

We use expressions (13) and (14) for dealing with shear strains distribution, $\gamma_a$, along the shaft radius in sectors, $H_1H_2$ and $H_3H_3$, respectively.

$$\gamma_{aH1H2} = \frac{\gamma_{H1H2}}{R_B} R, \tag{13}$$

$$\gamma_{aH2H3} = \frac{\gamma_{H2H3}}{R_B} R, \tag{14}$$

where

$$0 \le R \le R_B. \tag{15}$$

In formulas (13) and (14), $\gamma_{H1H2}$ and $\gamma_{H2H3}$ are the shear strains at the surface of the shaft in the two sectors.

We apply the following equations for determining $\gamma_{H1H2}$, $\gamma_{H2H3}$ and the torsion moments , $T_{H1}$ and $T_{H2}$, in sectors, $H_1H_2$ and $H_2H_3$, of the shaft:

$$\frac{\gamma_{H1H2}}{R_B} l_1 + \frac{\gamma_{H2H3}}{R_b} (l - l_1) = 0, \tag{16}$$

$$T_{H1} = \iint\limits_{(A)} \tau_{aH1H2} R dA, \tag{17}$$

$$T_{H2} = \iint\limits_{(A)} \tau_{aH2H3} R dA, \tag{18}$$

$$T_{H2} = T_{H1} - T_a, \tag{19}$$

where $\tau_{aH1H2}$ and $\tau_{aH2H3}$ are the stresses in the two sectors. Equation (16) is written by using the fact that the angle of twist of section, $H_1$, is zero (the angle of twist is expressed by the integrals of Maxell-Mohr). Equations (16), (17), (18) and (19) are solved by the MatLab.

We derive the unit damping energies, $\Delta u_{H1H2}$ and $\Delta u_{H2H3}$, by replacing of $\tau_a$ with $\tau_{aH1H2}$ and $\tau_{aH2H3}$ in formula (9), respectively. After that, the damping energy in the shaft is calculated by using (10), (11) and (12).

**3. Studying the Influence of Shaft Geometry, Loading, and Distribution of Material Parameters on Damping**

We illustrate the numerical results found-out by applying the approach for analyzing the damping energy developed in the previous section of this paper with the help of graphs shown in Fig. 2, Fig. 3, Fig. 4 and Fig. 5. When obtaining the numerical results, we assume that $l = 0.400$ m, $l_1 = 0.150$ m, $R_B = 0.005$ m and $T_a = 4$ Nm. The numerical results obtained show how the damping in the shaft under cyclic torsion is influenced by such factors as the shaft geometry, the magnitude and application point of loading, and the distribution of various material parameters in the shaft.

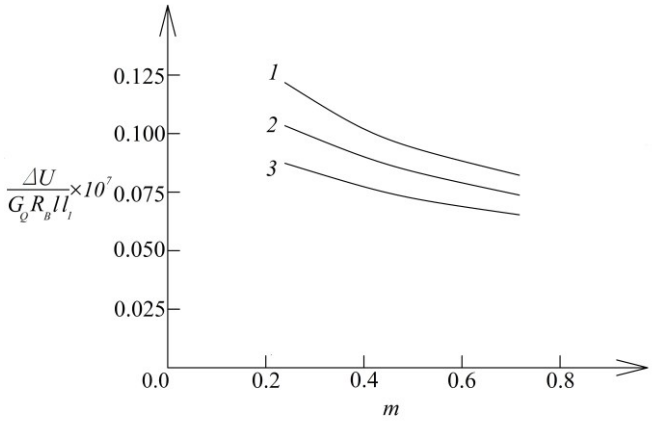


Fig. 2. The damping energy versus $m$ (1 - at $R_B = 0.005$ m, 2 - at $R_B = 0.006$ m and 3 - at $R_B = 0.007$ m).

Figure 2 displays the damping energy versus the parameter, $m$. Results for three values of $R_B$ are given in Fig. 2.

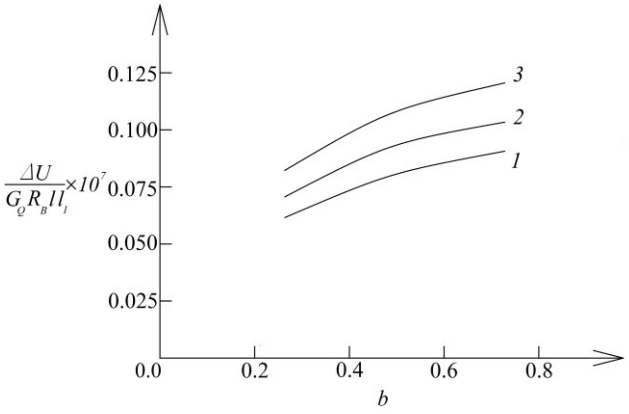


Fig. 3. The damping energy versus $b$ (1 - at $T_a = 4$ Nm, 2 - at $T_a = 5$ Nm and 3 - at $T_a = 6$ Nm).

As one can see the damping energy reduces with growth of $m$ (Fig. 1). This is an expected behavior since the shear modulus (and the shaft stiffness) increase with increase of $m$. It can also be seen in Fig. 2 that the growth of radius, $R_B$, causes reduction of the damping energy.

Figure 3 shows numerical results for the damping energy at increase of the parameter, $b$, and the torsion moment, $T_a$. It can be seen in Fig. 1 that the damping energy grows when $b$ increases. The increase of $T_a$ also causes growth of the damped energy in the shaft (Fig. 1).

Figure 4 illustrates the influence of the parameter, $p$, and the shaft length, $l$, on the damping energy.

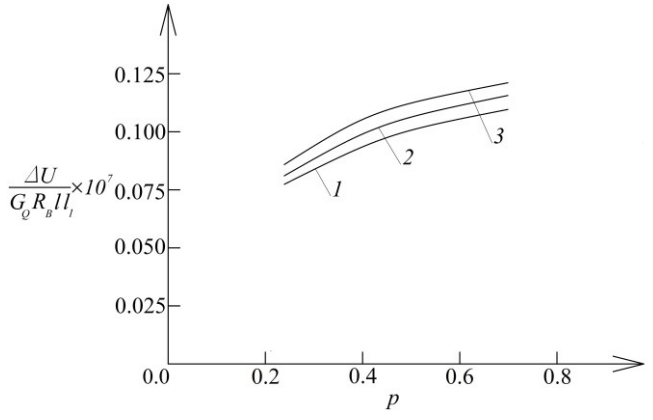


Fig. 4. The damping energy versus $p$ (1 - at $l = 0.400$ m, 2 - at $l = 0.500$ m and 3 - at $l = 0.600$ m).

The results shown in Fig. 4 indicate that increase of both $p$ and $l$ generate growth of the damping energy.

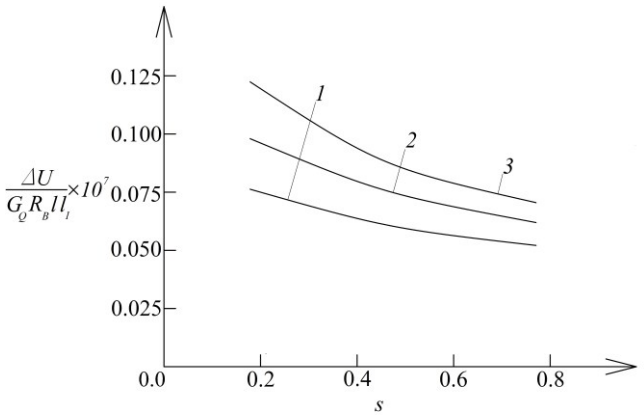


Fig. 5. The damping energy versus $s$ (1 - at $l/l_1 = 0.15$, 2 – at $l/l_1 = 0.30$ and 3 - at $l/l_1 = 0.45$).

Figure 5 displays the damping energy in the shaft versus the parameter, $s$, for three $l_1/l$ ratios (the $l_1/l$ ratio shows the location of the torsion moment along the shaft length). One can observe in Fig. 5 that the damping energy reduces with increase of $s$. Growth of $l_1/l$ ratio causes increase of the damping energy (Fig. 5).

**4. Conclusion**

The damping energy in a functionally graded viscoplastic shaft loaded in cyclic torsion is studied theoretically. The shaft is rigidly fixed in both ends, i.e. the shaft represents a one degree statically undetermined system. It is detected that:

1. The damping energy reduces with growth of the parameter, $m$.
2. The growth of radius, $R_B$, causes reduction of the damping energy.
3. The damping energy grows when the parameter, $b$, increases.
4. The increase of $T_a$ causes growth of the damped energy.
5. The increase of $p$ and $l$ generates growth of the damped energy.
6. The damping energy reduces with increase of the parameter, $s$.
7. The growth of $l_1/l$ ratio causes increase of the damping energy.

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