Analysis of Thin-walled Tri-layered Elastic-plastic Tubes in Loading and Unloading Phases

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https://doi.org/10.29227/IM-2025-02-02-040

Submission date: *2025-04-03* | Review date: *2025-04-24*

Abstract

Analysis of thin-walled tubes of elastic-plastic behavior is reported here. Hollow tubes are being increasingly used in the engineering. In most of the applications, tubes serve as pressure vessels. Therefore, the loading on tubes usually is pressure. As a result of this, the scientists and engineers mainly analyze the mechanical response of tubes to internal pressure. Most of the studies deal with tubes behaving linear-elastically. If, for instance, even at very low magnitudes of the pressure loading the tubes behave elastic-plastically, the equations for stresses obtained with linear-elastic behavior being assumed do not hold at all. In such situations, the tube response has to be analyzed by applying constitutive laws expressing the stress as a smooth non-linear function of the strain. The tubes analyzed here are tri-layered. The analysis is based on a theoretical formulation which incorporates loading as well as unloading phases of the response of the tubes. Formulas for the stresses in the layers are derived. The length of the tube in the hoop direction and the length of the tube radius are expressed as functions of strains in the phase of loading. A detailed study of the tube response during unloading is carried out. The residual stresses as well as the residual lengths of the tube in the hoop direction and the radius are presented as functions of the residual strains. Diagrams illustrating the functional relations between the characteristics of the tube response and the parameters of the tube geometry and pressure during loading and unloading phases are shown.

Keywords: *Tube, Elastic-plastic behavior, Pressure, Unloading*

1. Introduction

Various hollow tubes are being increasingly applied in many segments of the quickly developing engineering [1, 2]. Usually, in their numerous practical applications tubes work as pressure vessels [3, 4]. Therefore, the most frequent case of loading on tubes is the pressure. As a result of this, the attention of scientists and engineers around the globe usually is confined to the analysis of the mechanical response of tubes to internal pressure [5-7]. The intensive activities of the research community have resulted in deriving of useful expressions for stresses in tubes due to pressure loading [5-7]. Most of the studies treat tubes behaving linear-elastically under loading [3, 4]. For such tubes, closed-form equations for stresses are obtained. Applicability of these equations in design of tubes, however, is rather restricted by the principle assumption – this for linear-elastic behavior. If, for instance, even at very low magnitudes of the pressure loading the tubes behave elastic-plastically, the equations for stresses obtained with linear-elastic behavior being assumed do not hold at all. In such situations, the tube response has to be analyzed by applying constitutive laws expressing the stress as a smooth non-linear function of the strain.

Here, elastic-plastic response of tri-layered tubes under pressure loading is analyzed (as known, the layered materials exceed the homogeneous materials [8-15]). The tubes are thin-walled. Besides, the tubes behave elastic-plastically at any level of the pressure loading. This imposes the response of the tubes to be treated by application of a stress-strain dependency that is smoothly non-linear. A theoretical formulation for investigating the tube response in both loading and unloading phases is presented. A detailed description of an approach for determining of the residual length of the tube in the hoop direction, the residual radius and stresses as functions of the residual strains after unloading is given. Diagrams illustrating the functional relations between the characteristics of the tube response and the parameters of the tube geometry and pressure during loading and unloading phases are shown.

2. Theoretical formulation for analysis of the tube response

The radius of the tube in Fig. 1 is R. The tube is thin-walled, i.e. the wall thickness, h, is much smaller than R. The tube is made by three layers. The thickness of layers 1 and 3 is s (Fig. 1). Therefore, layer 2 has a thickness of $h-2s$. Layers 1 and 3 have identical material properties.

The materials of the tube layers have elastic-plastic behavior. Relationship (1) is employed for the stress-strain response of these materials (the response is presented in Fig. 2) [16].

$$\sigma_i = E_i\varepsilon - Q_i\varepsilon^{m_i}, \tag{1}$$

$$i = 1,2,3, \tag{2}$$

where σ_i and ε are the stress and strain, E_i, Q_i and m_i are parameters. Since layers 1 and 3 are identical,

$$E_3 = E_1, \tag{3}$$

$$Q_3 = Q_1, \tag{4}$$

$$m_3 = m_1. \tag{5}$$

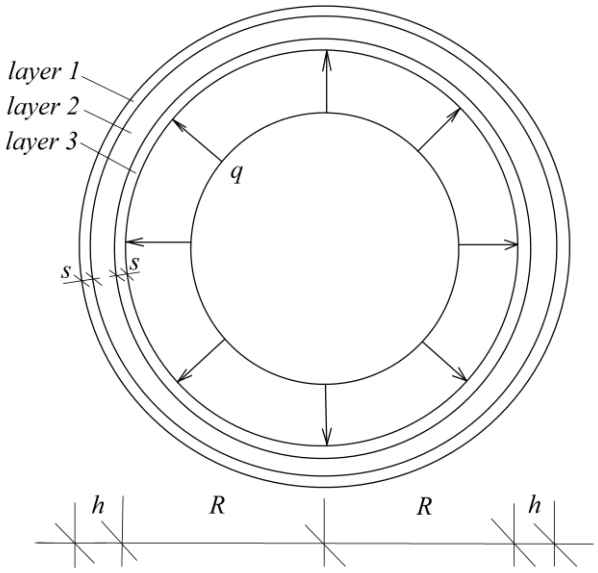

Fig. 1. Tri-layered tube.

Because the tube is thin-walled and layers 1 and 3 are identical, the hoop stresses in them are equal, i.e.

$$\sigma_3 = \sigma_1, \tag{6}$$

where the subscripts 1 and 3 refer to the corresponding layers.

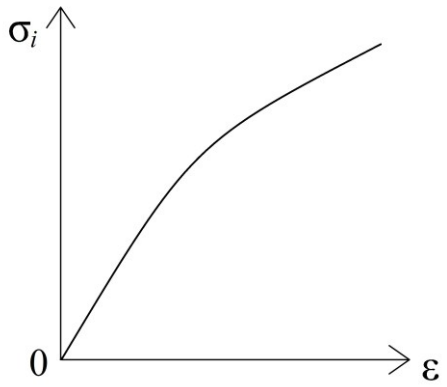

Fig. 2. Stress-strain curve.

The hoop force, N, in the wall is [16]

$$N = qR, \tag{7}$$

where q is the pressure in the tube (Fig. 1). Formula (7) is derived for a tube portion with unit length.

From the other hand, the hoop force can be expressed via the hoop stresses in the layers by equilibrium equation (8).

$$N = 2s\sigma_1 + (h - 2s)\sigma_2, \tag{8}$$

where σ_1 and σ_2 are the hoop stresses in layers 1 and 2, respectively. These stresses are non-linear functions of the strain (refer to (1)). Equation (8) is applied for obtaining of the strain, ε. Then, inserting of the strain in (1) yields the stresses in the layers.

The diagram of stresses is reported schematically in Fig. 3. Here, the stress in layer 1 is bigger than the stress in layer 2 because it is assumed that

$$E_1 > E_2, \tag{9}$$

$$Q_1 < Q_2. \tag{10}$$

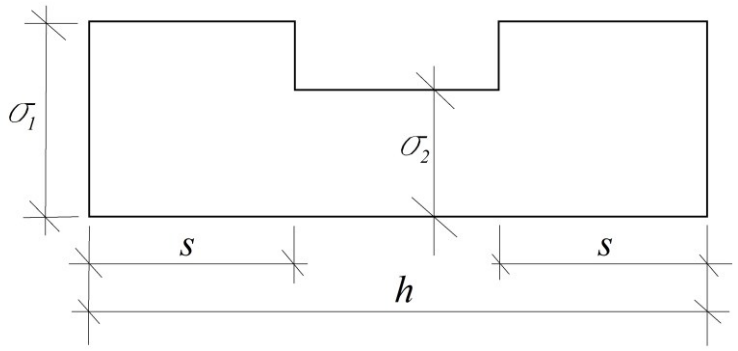

Fig. 3. Distribution of the stresses through the wall thickness.

The length, l_{df} , of the tube wall in the hope direction in deformed state is given exactly by

$$l_{df} = 2\pi R + \Delta, \tag{11}$$

where Δ is the increase of the wall length. The equation giving Δ is

$$\Delta = 2\pi R\varepsilon. \tag{12}$$

The solution for the radius, R_{df} , of the deformed tube is

$$R_{df} = R(1+\varepsilon). \tag{13}$$

One can confirm that the above analysis is right through the following consideration. At $Q_i = 0$ formula (1) yields the Hook's law. Besides, at $s = 0$ the tube is homogeneous. From this, it follows that at $Q_i = 0$ and $s = 0$ equations (8), (11), (12) and (13) should give the strain, the stress, the length of the deformed tube wall in hope direction, the increase of the wall length, and the radius of the deformed tube made of a homogeneous linear-elastic material. This is verified by letting $Q_i = 0$ and $s = 0$ in (8), (11), (12) and (13). The resulting solutions are

$$\sigma = \frac{qR}{A}, \tag{14}$$

$$\varepsilon = \frac{qR}{EA}, \tag{15}$$

$$l_{df} = 2\pi R + \frac{2\pi qR^2}{EA}, \tag{16}$$

$$\Delta = \frac{2\pi qR^2}{EA}, \tag{17}$$

$$R_{df} = R\left(1 + \frac{qR}{EA}\right), \tag{18}$$

where $A = h$ since a tube portion with unit length is treated. The exact correspondence of (14) – (18) with solutions for linear-elastic homogeneous tube in Ref. [16] demonstrates the rightness of the present analysis.

Determination of the residual strains and stresses in the tube wall after unloading, i.e. after removal of the pressure, is related to linear-elastic analysis of the tube behavior. The strain, ε_{el}, yielded by the linear-elastic analysis has to be subtracted from the strain derived by the non-linear equation (8) in order to determine the residual strains, ε_{res}, i.e.

$$\varepsilon_{res} = \varepsilon - \varepsilon_{el}. \tag{19}$$

The linear-elastic analysis of the tube is carried-out through equilibrium equation (20). This equation is used for obtaining of the strain, ε_{el}.

$$N = 2s\sigma_{1el} + (h - 2s)\sigma_{2el}, \tag{20}$$

where σ_{1el} and σ_{2el} are the linear-elastic stresses in the layers. These stresses are linear functions of the strain, i.e.

$$\sigma_{1el} = E_1\varepsilon_{el}, \tag{21}$$

$$\sigma_{2el} = E_2\varepsilon_{el}. \tag{22}$$

Therefore, an exact solution of (20) exists. This solution is

$$\varepsilon_{el} = \frac{qr}{2E_1 s + E_2(h - 2s)}. \tag{23}$$

Thus, the residual strain is

$$\varepsilon_{res} = \varepsilon - \frac{qr}{2E_1 s + E_2(h - 2s)}. \tag{24}$$

By using ε_{res}, the residual length, l_{res}, of the tube wall in the hoop direction after linear-elastic unloading can be calculated straightforward, i.e.

$$l_{res} = 2\pi R + 2\pi R\varepsilon_{res} \tag{25}$$

or

$$l_{res} = 2\pi R(1 + \varepsilon_{res}). \tag{26}$$

The residual length of the tube wall is related to the residual length of the radius, R_{res}, of the tube by (27).

$$l_{res} = 2\pi R_{res}. \tag{27}$$

The solution of (27) for R_{res} (with considering (26)) is

$$R_{res} = R(1 + \varepsilon_{res}). \tag{28}$$

The solutions for the residual stresses, σ_{1res} and σ_{2res}, in the tube layers after linear-elastic unloading are

$$\sigma_{1res} = \sigma_1 - \sigma_{1el}, \tag{29}$$

$$\sigma_{2res} = \sigma_2 - \sigma_{2el}. \tag{30}$$

Substituting (21) and (22) in (29) and (30), we have

$$\sigma_{1res} = \sigma_1 - E_1 \varepsilon_{el}\,, \tag{31}$$

$$\sigma_{2res} = \sigma_2 - E_2 \varepsilon_{el}\,. \tag{32}$$

3. Exploring the functional relations between the mechanical response and the tube parameters

The theoretical formulation for analysis of the elastic-plastic response of tri-layered thin-walled tubes in loading and unloading phases developed herein has a very good potential for detailed studies revealing specific characteristics which aids for a better understanding of the tube behavior. The paper, here, is focused on applications of the theoretical formulation for determining the functional relations between the mechanical response and the parameters of the tube. The values of the radius and the thickness of the tube used here are $R = 0.200$ m and $h = 0.002$ m, respectively. The diagrams of these functional relations are reported in non-dimensional form in Fig. 4 and Fig. 5.

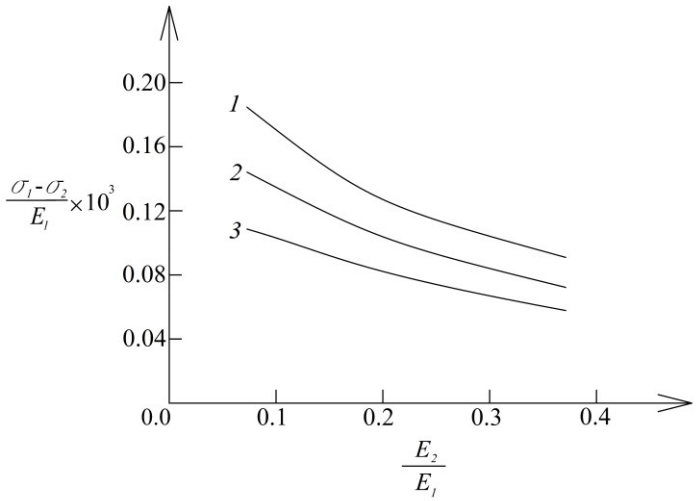

Fig. 4. The $(\sigma_1 - \sigma_2)/E_1$ - E_2/E_1 curves (1 – for $s/h = 0.05$, 2 – for $s/h = 0.10$ and 3 – for $s/h = 0.15$).

First, the tube response in the process of loading is determined.

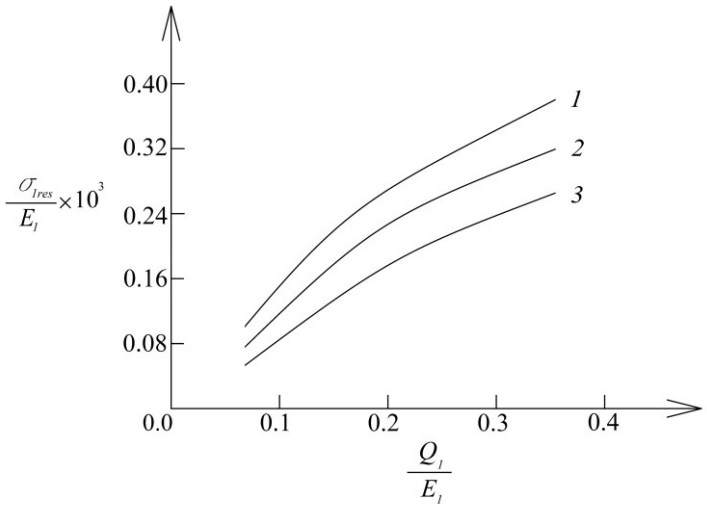

Fig. 5. The σ_{1res}/E_1 - Q_1/E_1 curves (1 – for $m_1 = 0.3$, 2 – for $m_1 = 0.5$ and 3 – for $m_1 = 0.7$).

With that end in view, the functional relation between the difference in the values of the stresses in layers 1 and 2 of the tube, i.e. $\sigma_1 - \sigma_2$, and E_2/E_1 ratio is studied. The difference, $\sigma_1 - \sigma_2$, is reported in non-dimensional form as $(\sigma_1 - \sigma_2)/E_1$. The functional relation between the difference in the values of the stresses in the layers and the thickness of layer 1 (the latter is reported in non-dimensional form as s/h) is also a subject of investigation. The diagrams of the functional relation, $(\sigma_1 - \sigma_2)/E_1$ - E_2/E_1, at different s/h ratios can be viewed in Fig. 4. One can infer from these diagrams that the difference in the values of the stresses in layers is getting smaller with growing of E_2/E_1 ratio (Fig. 4). One can infer also

that the growth of s/h ratios makes the difference in the values of the stresses in layers smaller (Fig. 4).

The process of unloading of the tube is also determined. In this connection, the residual stress, σ_{1res}, in layer 1 after removal of the pressure in the tube is studied. The residual stress is reported in non-dimensional form as σ_{1res}/E_1. The coupled effect of Q_1/E_1 ratio and the magnitude of the parameter, m_1, on the residual stress in layer 1 is studied. The diagrams of the corresponding functional relations are reported in Fig. 5. It is revealed from these diagrams that the residual stress is getting higher as the Q_1/E_1 ratio grows (Fig. 5). It is revealed also that as the magnitude of the parameter, m_1, becomes larger, the residual stress in layer 1 drops (Fig. 5).

4. Conclusion

The elastic-plastic response of a tri-layered thin-walled tube is analyzed for both loading and unloading phases. A theoretical formulation for the analysis is presented. Determination of the stresses and strains in the layers of the tube during loading and after unloading, i.e. after pressure removal, is detailed. The rightness of the investigation is demonstrated by the exact correspondence of the results of the elastic-plastic analysis at the boundary case $Q_i = 0$ and $s = 0$ with the results of a known solution for homogeneous linear-elastic tubes. The residual length of the tube in the hoop direction and the residual length of the radius of the tube after unloading are derived too. Functional relations between the mechanical response and the parameters of the tube are found. The relation between the difference in the values of the stresses in layers 1 and 2 and E_2/E_1 ratio indicates a continuous reduction of the difference. The latter reduces also with rise of s/h ratio. The relation between the residual stress in layer 1 and Q_1/E_1 ratio shows a continuous rise of the residual stress. The rise of the magnitude of the parameter, m_1, results in reduction of the residual stress in layer 1 of the tube.

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