Studying of a Functionally Graded Pressure Vessel Having Non-linear Viscoelastic Behavior

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Abstract

In this study, a functionally graded pressure vessel having form of a sphere is analyzed. It is known that pressure vessels are commonly used for keeping fluids. Many scientists and technical specialists in the area of engineering design and manufacturing are permanently engaged in research focused on enhancing the efficiency and safety of the pressure vessels. Application of modern materials represents an important step in the way towards modernization and raising the economic efficiency of the pressure vessels. Among the new engineering materials which have been introduced in the recent decades, the functionally graded composites show up with their excellent properties. The intense research in the area of technologies of these materials in many countries offer exceptional opportunities for making not only component parts of various devices, facilities and constructions but also for making of whole products including pressure vessels. The response of the functionally graded vessel studied here is non-linear viscoelastic. The analysis aims mainly to help for determining and understanding the relations between the vessel response and the parameters of the vessel viscoelastic model. The vessel response (i.e., the stresses in the vessel wall) is investigated. An approach that deals with equilibrium of an infinitesimal element of the vessel wall is detailed and applied for obtaining the stresses. The approach can be used for thin-walled pressure vessels with arbitrary symmetrical distribution of material properties in reference to the middle surface. Some typical results yielded by this approach are applied for building-up diagrams of the relations between the vessel response and the model parameters.

Keywords: *Pressure vessel, Sphere, Functionally graded material, Viscoelastic behavior*

1. Introduction

Pressure vessels are commonly used for keeping fluids in many industries and technical facilities [1, 2]. Enhancing the efficiency and safety of the pressure vessels is a permanent task for the scientists and specialists engaged in the research, design, manufacturing and exploitation of these vessels [3-5]. The practical significance of this task increases because failures of pressure vessels may cause not only collapse of structures and facilities with sizeable financial damages but also may lead to human lives loss. The advance in the science and technologies of pressure vessels requires not only development of new methods and approaches for analyzing the vessel response but also adopting of modern engineering materials for manufacturing of vessels. Using of such materials represents an important step in the way towards modernization and raising the economic efficiency of the pressure vessels. Among the various engineering materials which have been developed in the recent decades, the functionally graded materials show up with their excellent properties [6-9]. The high achievements in the development of technologies of these materials in many countries in the world offer exceptional opportunities for making not only component parts of various devices, facilities and constructions but also for making of whole products [10-13].

The current study is engaged with analysis of the response of a functionally graded closed pressure vessel. The vessel has the form of a thin-walled sphere. The properties of the material that is used for building-up the vessel have a symmetrical distribution in reference to the vessel middle surface. Since the behavior of certain materials is non-linear viscoelastic, there is a growing interest in the academic circles especially in dealing with actual response of pressure vessels made by such construction materials. This problem, however, is notable for its complexity as the physical law that relates the stress, strain and time is non-linear. Here, a tri-element non-linear model of a viscoelastic solid is applied in analyzing the vessel response. The latter is presented by the stresses in the vessel wall. The necessary equations for predicting these stresses are derived by applying an approach whose theoretical base is the equilibrium of an infinitesimal element of the vessel wall. The approach is general in the sense that it can be used for dealing with thin-walled pressure vessels with arbitrary symmetrical distribution of material properties in reference to the middle surface. Relations between the vessel response and various parameters of the model are found and their diagrams are reported.

2. Analysis of the vessel response

A closed pressure vessel in the form of sphere is presented in Fig. 1. The two geometrical parameters of this vessel, namely the radius and the thickness are designated by R and h, respectively (Fig. 1). The vessel wall is thin, i.e. $h << R$. A functionally graded material is used for the wall. Besides, the properties of this material have a symmetrical distribution in reference to the middle surface of the vessel. Therefore, the meridional and the hoop stresses, σ_m and σ_p, in the wall also have a symmetrical distribution in reference to the middle surface, i.e. the stresses are symmetric functions of the coordinate x, where x is directed along the wall thickness (the origin of the x – axis is on the middle surface of the vessel wall). The equilibrium of an infinitesimal element of the vessel can be analyzed via equation (1).

$$qds_1ds_2 - \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_m(x)dxds_2d\theta - \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_p(x)dxds_1d\varphi = 0, \tag{1}$$

where

$$\sigma_m = \sigma_m(x), \tag{2}$$

$$\sigma_p = \sigma_p(x), \tag{3}$$

$$-\frac{h}{2} \le x \le \frac{h}{2}, \tag{4}$$

$$d\theta = \frac{ds_1}{\rho_m}, \tag{5}$$

$$d\varphi = \frac{ds_2}{\rho_p}. \tag{6}$$

The quantity, q, in (1) is the pressure in the vessel.

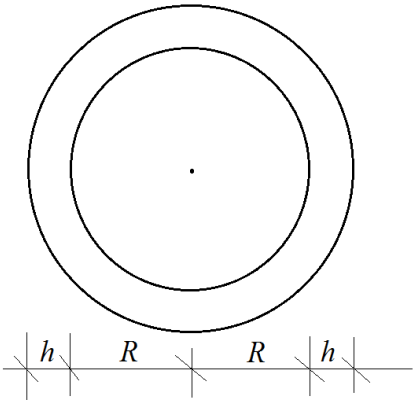

Fig. 1. Vertical section passing through the centre of the vessel (geometrical parameters).

Before continuing further with the analysis, validity of equation (1) is confirmed. This is done by checking if equation (1) converts into a known equation for homogeneous thin-walled vessels [5]. If the vessel for which equation (1) is obtained is homogeneous, then the solutions of the two integrals involved in (1) are

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_m(x)dx = \sigma_m, \tag{7}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_p(x)dx = \sigma_p \tag{8}$$

and equation (1) converts into

$$qds_1ds_2 - \sigma_m ds_2 d\theta - \sigma_p ds_1 d\varphi = 0, \tag{9}$$

which corresponds exactly to the equation for homogeneous vessel in [5].

There is symmetry in the vessel in Fig. 1. Thus, one has

$$\sigma_m = \sigma_p = \sigma, \tag{10}$$

$$\rho_m = \rho_p = R \tag{11}$$

and equation (1) simplifies to

$$q - \frac{2}{R}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma dx = 0\,. \tag{12}$$

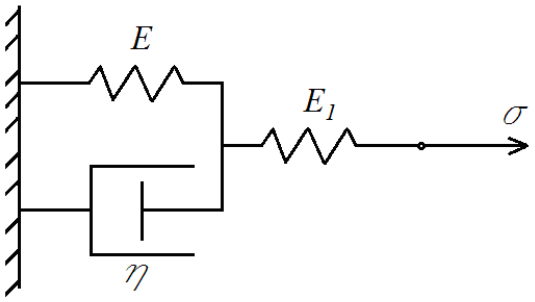

Fig. 2. Tri-element non-linear model of a viscoelastic solid.

Since the vessel response to the pressure is non-linear viscoelastic, the stress, σ, is related to the strain, ε, in the wall through the non-linear model of a viscoelastic solid presented in Fig. 2. The linear elastic and the linear viscoelastic components of the response are modeled by a linear spring, E, and a linear dashpot, η, respectively (Fig. 2). For modeling the non-linear elastic component of the response, a non-linear spring, E_1, is added (Fig. 2). Here, E_1 is a non-linear function of ε. This function is presented by formula (13).

$$E_1 = \frac{E_0}{1+\delta\varepsilon}, \tag{13}$$

where δ is a parameter ($\delta > 0$).

The relationship between the stress, σ, and the strain, ε_1, in the non-linear spring is

$$\sigma = E_1\varepsilon_1\,. \tag{14}$$

The response of the linear spring and the linear dashpot to the stress, σ, is given by

$$\varepsilon_2 = \frac{\sigma}{E}\left(1 - e^{-\frac{Et}{\eta}}\right), \tag{15}$$

where ε_2 is the strain in the linear spring, t is the time.

The strain in the model is given by adding the strains in the non-linear and the linear springs

$$\varepsilon = \varepsilon_1 + \varepsilon_2\,. \tag{16}$$

Combining the equations (13), (14), (15) and (16) gives the non-linear relationship between the stress, strain and time, i.e.

$$\varepsilon = \sigma\left[\ \frac{1}{E}\left(1 - e^{-\frac{Et}{\eta}}\right) + \frac{1+\delta\varepsilon}{E_0}\ \right]. \tag{17}$$

The non-linear viscoelastic response of the vessel to the pressure is presented by (17). The validity of formula (17) is confirmed below. At

$$\delta = 0 \tag{18}$$

formula (17) converts into

$$\varepsilon = \sigma\left[\ \frac{1}{E}\left(1 - e^{-\frac{Et}{\eta}}\right) + \frac{1}{E_0}\ \right], \tag{19}$$

which coincides with the relation between the stress, strain and time of the Kelvin model of a linear viscoelastic solid [14] (this coincidence confirms the validity of (17) since at $\delta = 0$ the response of the spring, E_1, gets linear (refer to (13) and(14)) and the non-linear model in Fig. 2 converts into the Kelvin model).

It was specified above that the properties of the material have a symmetrical distributions in reference to the middle surface of the vessel. These distributions are given by

$$\eta = \eta_c + \frac{\eta_v - \eta_c}{\left(\frac{h}{2}\right)^{\lambda_1}} |x|^{\lambda_1}, \tag{20}$$

$$E = E_c + \frac{E_v - E_c}{\left(\frac{h}{2}\right)^{\lambda_2}} |x|^{\lambda_2}, \tag{21}$$

$$E_0 = E_{0c} + \frac{E_{0v} - E_{0c}}{\left(\frac{h}{2}\right)^{\lambda_3}} |x|^{\lambda_3}, \tag{22}$$

$$\delta = \delta_c + \frac{\delta_v - \delta_c}{\left(\frac{h}{2}\right)^{\lambda_4}} |x|^{\lambda_4}, \tag{23}$$

where

$$-\frac{h}{2} \le x \le \frac{h}{2}. \tag{24}$$

In formulas (21), (22) and (23), the subscripts, c and v, refer to the middle and outer/inner surfaces of the vessel wall, respectively, λ_1, λ_2, λ_3 and λ_4 are parameters of the distributions.

Equations (9) and (14) are combined and solved for ε by the MatLab.

3. Parametric Investigation

One of the main purposes of this study is to develop an analysis that helps to understand the relation between the response of the pressure vessel and the distribution of the properties of the non-linear viscoelastic model across the wall of the vessel. If this relation is fully understood then researchers and engineers dealing with pressure vessels may use it in their work. In order to understand the above mentioned relation, the analysis of the functionally graded pressure vessel response is applied here to carry out a detailed investigation by varying the parameters of the theoretical model. Then the results generated by the investigation are used for building-up diagrams which give a clear idea for the relation between the vessel response and the properties distribution (these diagrams are presented here in Figs. 3, 4 and 5). A vessel with radius $R = 0.6$ m and wall thickness $h = 0.005$ m is analyzed. The vessel response is presented by the stresses in the middle and outer/inner surfaces of the vessel wall (the stresses in the outer and the inner surfaces of the wall are the same because of the symmetrical distributions of the properties of the viscoelastic model across the wall (refer to formulas (15), (16) and (17)). The stresses and most of the other parameters used for building-up the diagrams in Figs. 3, 4 and 5 are reported as non-dimensional quantities.

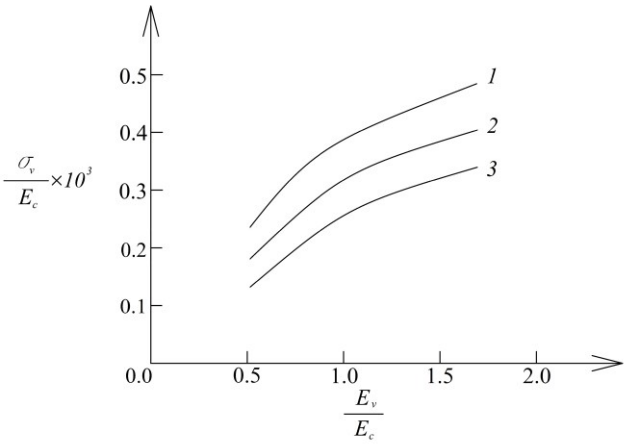

Fig. 3. Diagrams of the relation between the non-dimensional stress on the outer surface of the vessel wall, σ_v / E_c, and the ratio of the values of E on the outer and the middle surfaces of the wall, E_v / E_c, (1 – at $\delta_v / \delta_c = 0.5$, 2 – at $\delta_v / \delta_c = 1.0$ and 3 – at $\delta_v / \delta_c = 2.0$).

The diagrams reported in Fig. 3 help us to understand the relation between the non-dimensional stress on the outer surface of the wall, σ_v / E_c, and the ratio of the values of E on the outer and the middle surfaces of the vessel wall, E_v / E_c.

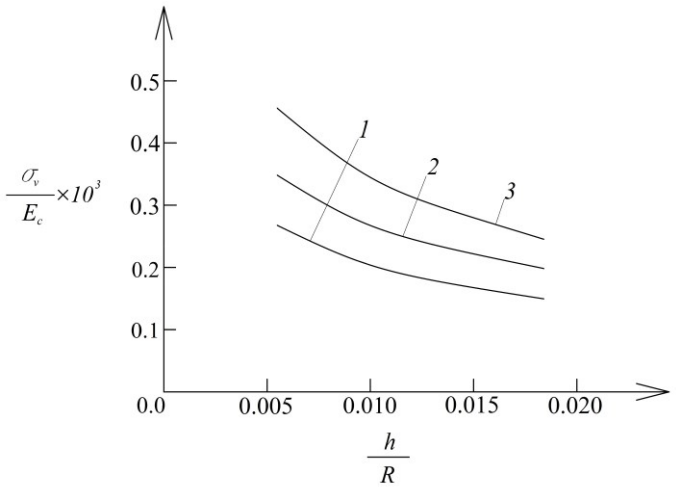

Fig. 4. Diagrams of the relation between the non-dimensional stress on the outer surface of the vessel wall, σ_v / E_c, and h / R ratio (1 – at $E_{0v} / E_{0c} = 0.5$, 2 – at $E_{0v} / E_{0c} = 1.0$ and 3 – at $E_{0v} / E_{0c} = 2.0$).

The manner in which this relation is influenced by the ratio of the values of δ on the outer and the middle surface of the vessel, δ_v / δ_c, also is clarified (the diagram of the relation between σ_v / E_c and E_v / E_c is built-up for three δ_v / δ_c ratios in Fig. 3). One can observe from Fig. 3 that the non-dimensional stress in the outer surface of the vessel wall is influenced in a significant degree by both E_v / E_c and δ_v / δ_c ratios. The shape of the diagrams in Fig. 3 indicates that the non-dimensional stress in the outer surface of the wall can be kept low by reducing E_v / E_c ratio. Another conclusion that can be made by observing the diagrams in Fig. 3 is that the non-dimension stress can be kept low also by increasing δ_v / δ_c ratio.

The relation between the non-dimensional stress in the outer wall of the pressure vessel, σ_v / E_c, and h / R ratio can be understood by inspecting the diagrams reported in Fig. 4. These diagrams make clear also the relation between σ_v / E_c and the ratio of the values of E_0 in the outer and the middle surface of the vessel wall, E_{0v} / E_{0c}. One of the manners for keeping the non-dimensional stress in the outer surface of the wall low is to increase h / R ratio (Fig. 4). Viewing the diagrams in Fig. 4 clarifies also that the non-dimensional stress can be kept low by reducing E_{0v} / E_{0c} ratio.

The investigation generates results that can help also to understand the relation between the non-dimensional stress in the middle surface of the vessel wall, σ_c / E_c, and the ratio of the values of η in the outer and middle surfaces of the wall, η_v / η_c.

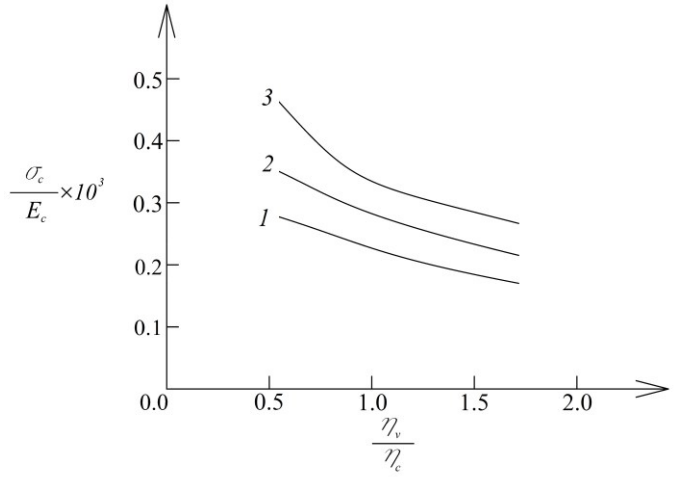

Fig. 5. Diagrams of the relation between the non-dimensional stress on the middle surface of the vessel wall, σ_c / E_c, and the ratio of the values of η on the outer and the middle surfaces of the wall, η_v / η_c, (1 – at $R = 0.60$ m, 2 – at $R = 0.75$ m and 3 – at $R = 0.90$ m).

The diagrams of this relation are built-up in Fig. 5. The effect of R on the relation between the non-dimensional stress in the middle surface of the vessel and η_v / η_c ratio can also be understood from the diagrams built-up in Fig. 5. As can be seen from the diagrams in Fig. 5 the non-dimensional stress can be kept low by reducing η_v / η_c ratio.

4. Conclusion

This work presents an analysis of a functionally graded closed pressure vessel in the form of a sphere having non-linear viscelastic behavior. The results generated by the analysis help for understanding the relations between the vessel response and the properties and parameters of the theoretical model. The main conclusions from the analysis are:

1. The stress, σ_v, on the outer surface of the vessel wall can be kept low by reducing E_v / E_c ratio.
2. The stress, σ_v, can be kept low also by increasing the δ_v / δ_c ratio.
3. Increasing of h / R ratio also contributes for keeping low values of the stress, σ_v.
4. Reducing E_{0v} / E_{0c} ratio also has a significant contribution for keeping low values of the stress, σ_v.
5. The value of the stress, σ_c, on the middle surface of the vessel wall can be kept low by enhancing η_v / η_c ratio.

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