Application of Probabilistic-Based Weights in Precise Levelling Adjustments

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Abstract

The selection of observation weights plays a critical role in the reliability of geodetic network adjustment, particularly when the true measurement accuracy is unknown. In such situations, weights are commonly assigned based on assumed accuracy-related factors. In precise levelling, uncertainty is typically modeled as a function of distance, resulting in weights inversely proportional to levelling length; however, this assumption may not be optimal. This study investigates an alternative weighting strategy by adjusting the Third Levelling of Bulgaria (1975–1984) using probabilistically derived weights based on the Pearson distribution system. An iterative reweighting procedure was implemented, whereby the weights of line height differences were updated according to the magnitude of height corrections obtained in the preceding iteration. A total of 100 iterations were performed to assess the behavior of the reweighting process. The results exhibit a repetitive pattern, indicating a lack of convergence. Nevertheless, stable and nearly identical solutions were identified at the 10th, 39th, and 72nd iterations. These iterations were selected as they yielded minimal sums of adjusted benchmark standard errors, while the differences in adjusted benchmark heights relative to the preceding iterations remained below 0.02 mm. A comparison between the classical adjustment and the probabilistic approach at the tenth iteration demonstrates a substantial improvement in accuracy. The mean benchmark standard error decreased from 13.65 mm to 9.65 mm, with consistently lower standard errors observed for all benchmarks under the probabilistic scheme. Statistical testing using t-tests confirms that this improvement is significant at a confidence level exceeding 99%. Additionally, the classical adjustment produced systematically higher adjusted heights for 31 out of 32 benchmarks, with differences ranging from −6.8 mm to 21.4 mm. However, the results obtained using probabilistic weights are ambiguous with respect to the convergence stopping criterion. Moreover, compared with the results from the all-combination adjustment approach, the probabilistic weighting yields less plausible outcomes than the classical network adjustment.

Keywords: *Maximum Likelihood, Probability Theory, Probabilistic Weights, Precise Levelling, Levelling Accuracy*

1. Introduction

For more than a century, the most popular weights in the adjustment of precise levelling networks have been given by equation (1) [1-3].

$$W = \frac{constant}{L} \tag{1}$$

In equation (1), L denotes the length of the levelling line in kilometers. The constant in the nominator in (1) is chosen in such a way that the weights W are spread around the value 1. The weights are formed under the assumption that the uncertainty in the geometric levelling measurements is a function of the levelling distances. However, this hypothesis is at least disputable because it is not supported by the modern statistical methods [4-6]. As a result, various weighting approaches produce a smaller standard error of the adjusted benchmark heights, implying a lower degree of uncertainty.

Since the weighting scheme plays a crucial role in the reliability of adjustment results, the search for a robust and faithful approach remains ongoing. One promising direction is the application of observation weights derived from the maximum likelihood principle [7–9]. It is easy to prove that the weight of the i^{th} observation is a function of its random error v_i, the value of the curve of the probability density function of the distribution of all observations $f(v)$ in v_i, and the first derivative of $f'(v_i)$. Thus, the weight of the i^{th} observation can be written as in equation (2).

$$W_i = f'(v_i)/(f(v_i).v_i) \tag{2}$$

The ratio $f'(v_i)/f(v_i)$ can be given as the influence function $\Omega(v)$ of Mp estimation and presented by the equation (3) [7].

$$\Omega(v) = -\frac{(c_0+3c_2)v-\sigma c_1}{\sigma^2 c_0-\sigma c_1 v+c_2 v^2} \tag{3}$$

$$c_0 = 4\beta_2 - 3\beta_1 \tag{4}$$

$$c_1 = \sqrt{\beta_1}(\beta_2 + 3) \tag{5}$$

$$c_2 = 2\beta_2 - 3\beta_1 - 6 \tag{6}$$

The notations β_1 and β_2 are the skewness and the kurtosis of the distribution of the residuals v, calculated by Pearson's central moments k = {2, 3, and 4} by equations (7) – (9).

$$\beta_1 = \frac{m_3}{m_2^{3/2}} \tag{7}$$

$$\beta_2 = \frac{m_4}{m_2^2} \tag{8}$$

$$m_k = \frac{1}{n}\sum_{j=1}^{n}\left(v_j - \bar{v}\right)^k \tag{9}$$

The notation σ is Pearson's second central moment m_k. Suppose a symmetric unimodal distribution of residuals v. Thus, skewness $\beta_1 = 0$, and equation (3) turns into (10).

$$\Omega(v) = -\frac{(5\beta_2 - 9)v}{2\sigma^2\beta_2 + (\beta_2 - 3)v^2} \tag{10}$$

According to the theory of M-Estimation [10, 11], the weights (2) can be given by equation (11) [9].

$$W_i = -\frac{\Omega(v_i)}{v_i} = \frac{5\beta_2 - 9}{2\sigma^2\beta_2 + (\beta_2 - 3)v_i^2} \tag{11}$$

The primary aim of this study is to examine the behaviour of weights (11) in the adjustment of a real state levelling network and to compare these results with those obtained from a classical adjustment using weights (1). A secondary aim is to assess the reliability of the results produced by iterative reweighted least squares with weights (11).

2. Used data and their processing

In this study, the height differences from Bulgaria's Third Levelling are used in the upward direction, downward direction, and as their averages, corrected only by the mean staff (lath) meter corrections. The data are presented in Tables 1 and 2 in the study [5], while the network configuration is shown in Fig. 1.

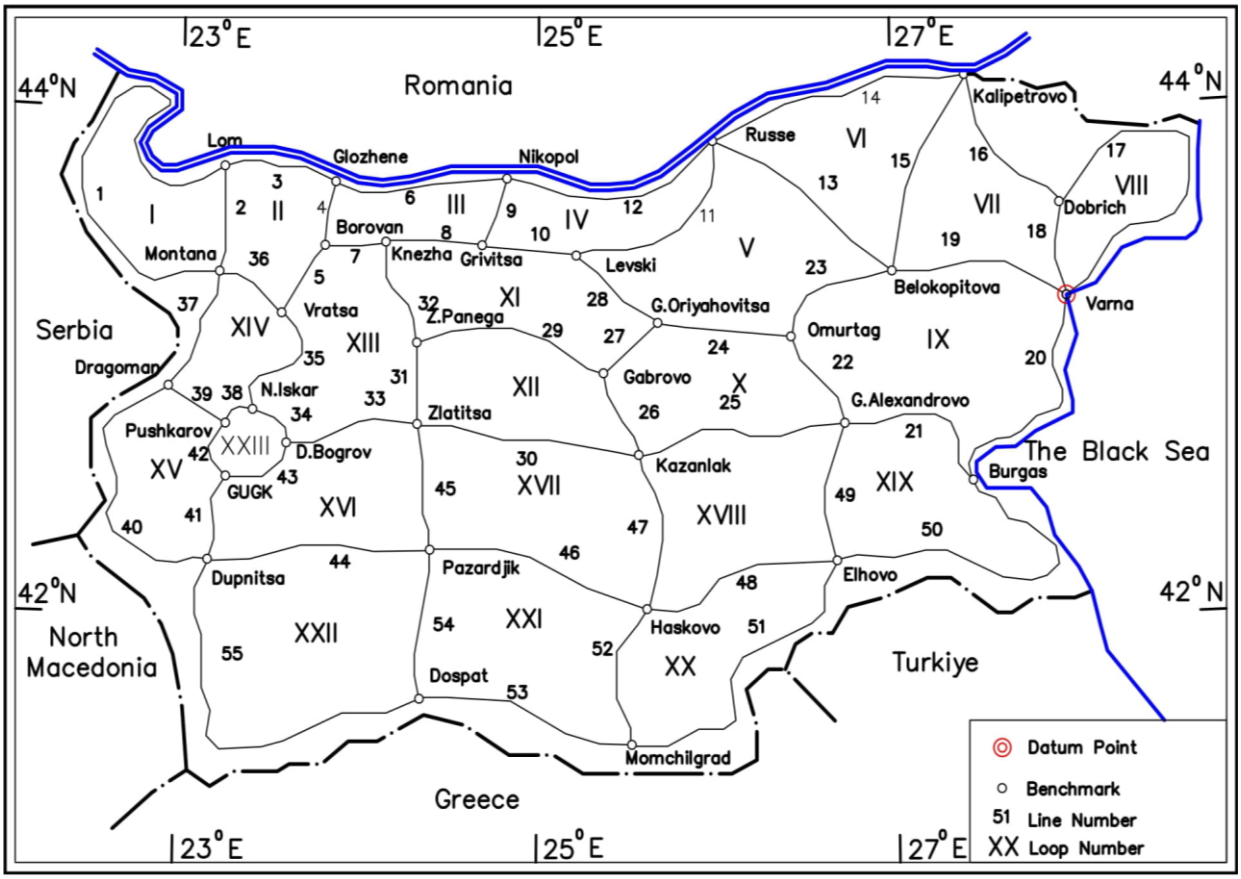

Fig. 1. The Third Levelling of Bulgaria Network /1975-1984/.

The network was adjusted using two different approaches. In the first, a classical adjustment was carried out, with weights defined according to equation (1). In the second, an iterative reweighting adjustment was performed using weights (11). To examine the convergence behaviour, 100 iterations were executed, although the differences in the adjusted benchmark heights between the 9th and 10th iterations were already below 0.02 mm. As a datum point in both adjustments, the century benchmark 28 in Varna was used.

3. Results

Figure 2 shows the distribution of standard errors of the adjusted benchmark heights over the *Mp* adjustment iterations up to the 100th iteration. Figure 3 compares the differences in adjusted benchmark heights at the 10th, 39th, and 72nd iterations. Figure 4 presents the differences between benchmark heights obtained from the classical adjustment and those derived from the *Mp* approach at the same iterations. Figure 5 illustrates the accuracy differences between the classical and probabilistic methods. Table 1 summarizes the results of a paired-samples t-test comparing the standard errors from the classical approach with those from the 10th iteration of the Mp approach.

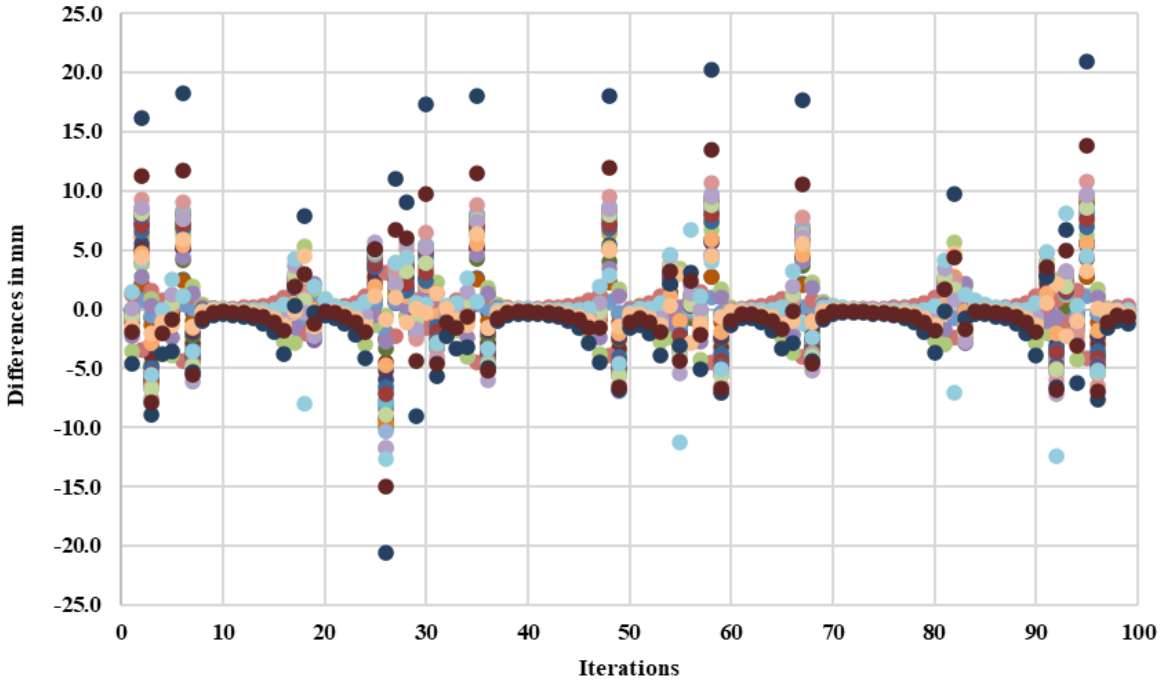

Fig. 2. Differences in the adjusted benchmark heights in mm, obtained in two sequential iterations.

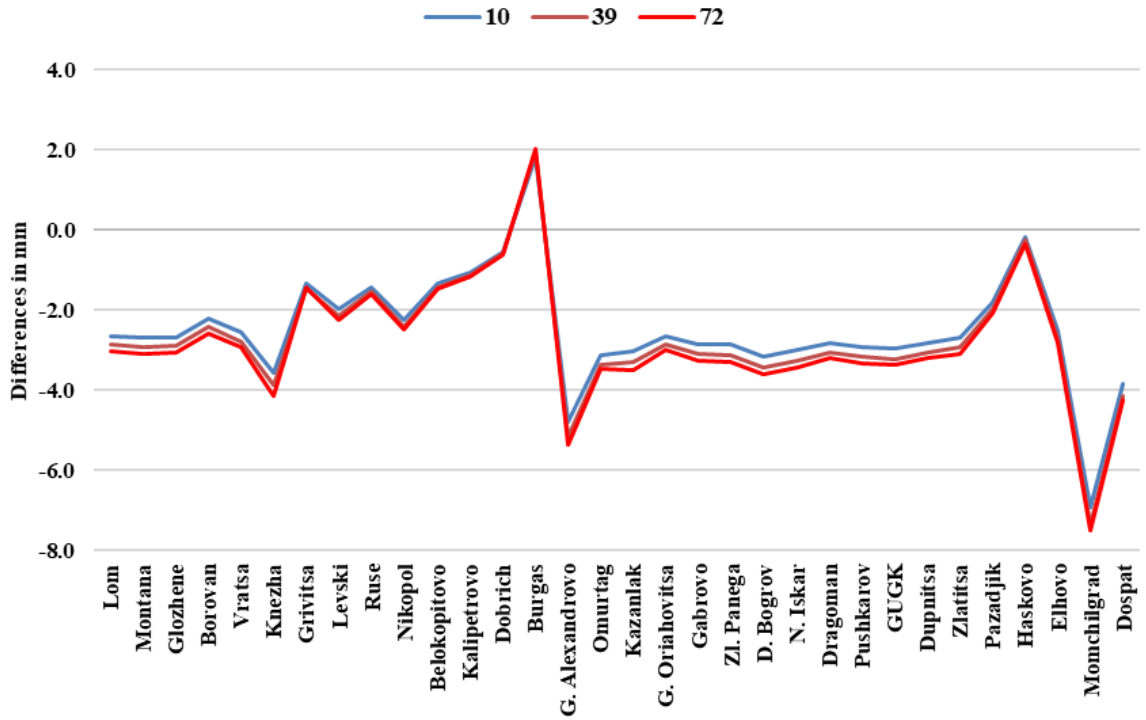

Fig. 3. Differences in the adjusted benchmark heights in mm, obtained in the first iteration and iterations 10, 39, and 72.

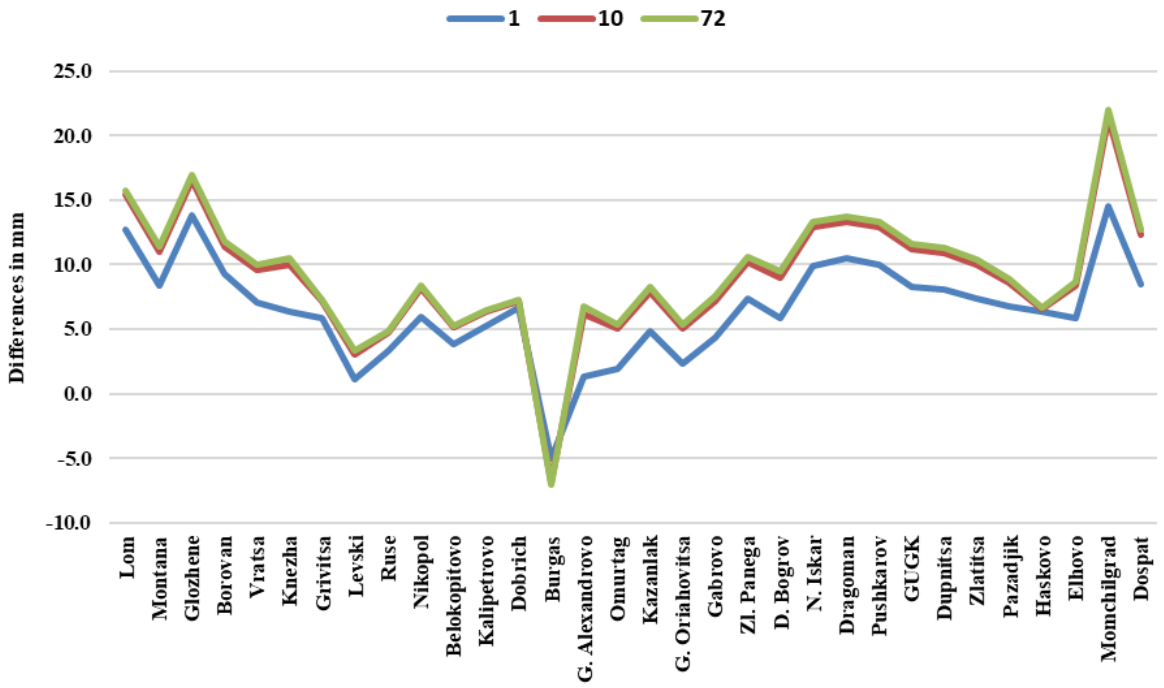

Fig. 4. Differences in the adjusted benchmark heights in mm, obtained in the classical adjustment and in iterations 10, 39, and 72 in the probabilistic-based weighting approach.

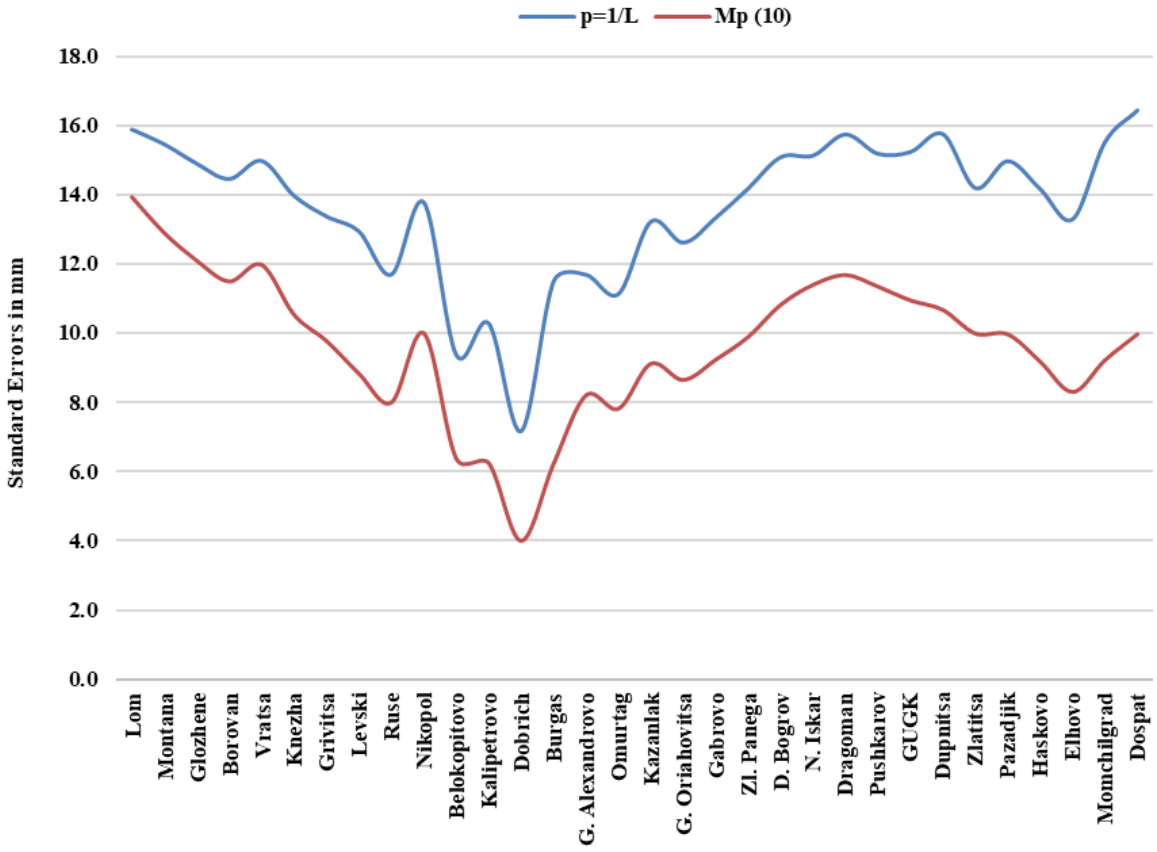

Fig. 5. Standard errors of the adjusted benchmark heights in mm, obtained by the classical adjustment and the probabilistic-weighted approach in the 10th iteration.

Tab. 1. Paired t-Test of mean benchmark standard errors, obtained in the classic and the probabilistic approach.

Statistic	Classic	Mp(10)
Mean	13.65	9.65
Variance	4.38	4.44
Observations	32	32
Pearson Correlation	0.889	
Hypothesized Mean Difference	0	
df	31	
t Stat	22.884	
P(T<=t) two-tail	**5.6E-21**	
t Critical two-tailed	2.040	

According to the results shown in Table 1, the accuracy obtained at the 10th iteration of the Mp approach is significantly higher than that achieved through the classical adjustment, with this conclusion supported at a confidence level greater than 99.99%.

4. Discussion

An analysis of the Mp adjustment results shown in Fig. 2 reveals the following:

- There is no clear convergence of the iterative process. The differences between adjusted benchmark heights across iterations fluctuate and do not converge even after 100 iterations. An initial reduction in differences up to the 10th iteration is followed by expansion until the 19th iteration, after which the pattern repeats with alternating contraction and expansion.
- Certain iterations, such as the 37th, 50th, and 69th, yield very similar results, indicating a cyclic behaviour. This suggests that convergence is unlikely even with many additional iterations, leading to ambiguity in selecting the most appropriate solution, as no single iteration is clearly optimal.
- Some iterations, notably the 10th, 39th, and 72nd, produce very similar adjusted benchmark heights. Moreover, the differences compared to the preceding iterations (9th, 38th, and 71st) are below 0.02 mm. However, using such a criterion to define a stopping point remains somewhat subjective, even though Fig. 3 indicates that the 39th and 71st iterations are closer to each other than the 10th and 39th iterations.

Further analysis of Fig. 6 and Table 1 shows that the root-mean-square errors of the adjusted benchmark heights obtained from the classical adjustment are higher than those from the 10th iteration of the probabilistic weighting approach (11). The differences in internal accuracy are statistically significant at a confidence level of $\alpha = 0.0001$, indicating with 99% confidence that the probabilistic weights yield more reliable results. However, this conclusion should be considered with caution, as the method also presents notable limitations.

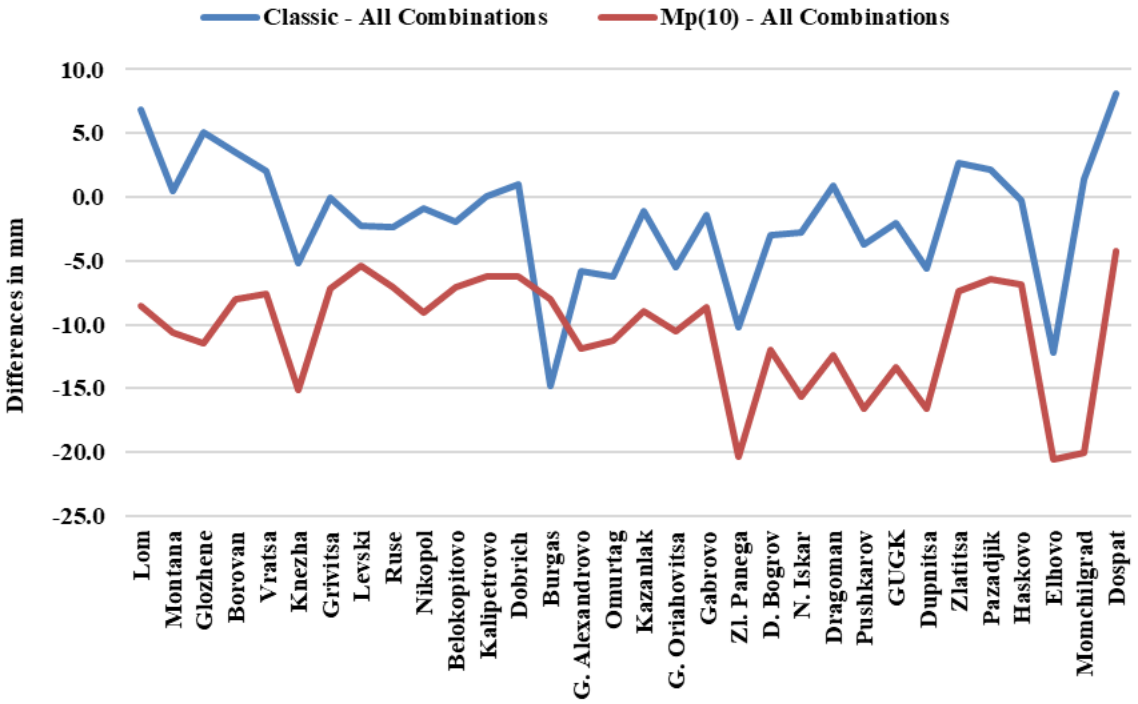

Fig. 6. Standard errors of the adjusted benchmark heights in mm, obtained by the classical adjustment and the probabilistic-weighted approach in the 10th iteration.

When real observations are used, assessing the reliability of adjustment results is often challenging. In such cases, simulation studies provide a valuable means of evaluating different adjustment methods. According to [6], the most robust and reliable approach

for levelling network adjustment is the all-combination method, or, specifically for levelling, the 3n combination method. This method is recommended when the ratio between the a posteriori accuracy of the "worst" and "best" observation sets exceeds 1.5.

For the Third Levelling of Bulgaria, this ratio is $(\sigma_worse/\sigma_better) = 6.24$, and thus $(\sigma_worse/\sigma_better)^2 \approx 39$. This result strongly supports the selection of the 3^n combination method over classical adjustment, M-estimation (with various objective functions), S-estimation (with different breakdown points), and MM-estimation. Under such conditions, the 3^n combination method yields results that are approximately 2–3 times more reliable than the alternative approaches, in addition to providing superior a posteriori accuracy.

Fig. 6 illustrates the reliability of the adjusted benchmark heights obtained from both the classical adjustment with weights (1) and the probabilistic iterative reweighting approach with weights (11), compared to the reference results from the 3^n combination method. A strong correlation is observed between the compared approaches; however, a systematic offset of about 9 mm on average is evident. The results from the classical adjustment are generally randomly distributed around those from the 3n method, except for the benchmarks Burgas, Elhovo, and Zlatna Panega, and exhibit a slight negative bias of approximately 1.7 mm on average.

In contrast, the results from the probabilistic approach show a pronounced systematic deviation from those of the 3n combination method, with an average offset of about 10.7 mm. This raises concerns regarding the reliability of the probabilistic iterative reweighting procedure using weights (11). Nevertheless, further detailed investigation is required to fully assess and explain this behaviour.

5. Conclusion

The study demonstrates that the choice of observation weights significantly influences the reliability and accuracy of geodetic network adjustment. The application of probabilistically derived weights based on the Pearson distribution, implemented through an iterative reweighting procedure, led to a notable improvement in accuracy compared to the classical distance-based weighting scheme. In particular, the tenth iteration provided substantially lower standard errors for all benchmarks, with statistical testing confirming the significance of this improvement.

However, the iterative process revealed a lack of clear convergence, as indicated by the cyclic behavior of the results and the absence of a well-defined stopping criterion, even though several iterations (notably the 10th, 39th, and 72nd) yielded stable and nearly identical solutions. Furthermore, when compared to the all-combination adjustment method, the probabilistic weighting produced less plausible results, raising concerns about its overall reliability.

In summary, while probabilistic weighting shows potential for enhancing adjustment accuracy, its instability and convergence ambiguity suggest that further refinement is necessary before it can be considered a robust alternative to classical weighting methods.

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